Groupoidification and 2-Linearization: Discrete and Smooth

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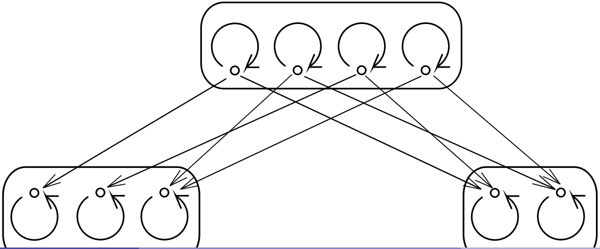


Motivation

Categorify a quantum mechanical description of states and processes.

	Sets	Categories
Classical	S: A set whose elements are *states* of a system	X: A groupoid with: •Ob: *states* •Mor: *symmetries* of states
Quantum	$L^2(S)$: Vector space of states (in fact, Hilbert space)	$\Lambda(X)$: 2-vector space of states (in fact, 2-Hilbert space)

We also can contrast classical and quantum *processes*: Each process has a "start" and "end" point: processes can be described by *spans*.
At first we will ignore any special structure to S (or X).

I will describe the category $Span_1(\mathbf{Gpd})$ and 2-category $Span_2(\mathbf{Gpd})$, and:

- "Degroupoidification", a functor $D : Span_1(\mathbf{Gpd}) \rightarrow \mathbf{Vect}$
- "2-linearization", a 2-functor $\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$

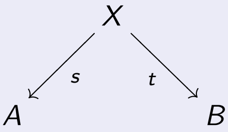
Both of these generalize an obvious "linearization" functor

$$L : Span_1(\mathbf{Set}) \rightarrow \mathbf{Vect}$$

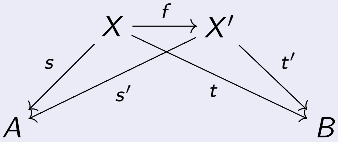
At first we will assume all groupoids mentioned are finite.

Definition

A **span** in a category **C** is a diagram of the form:



A *span map* f between two spans consists of a compatible map:



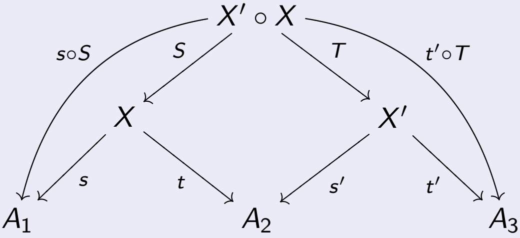
A *cospan* is a span in $\mathbf{C}^{\mathrm{op}}$.

If **C** is a category with pullbacks and terminal object I (hence all products), we can define:

Definition

The category $Span_1(\mathbf{C})$ has:

- **Objects**: Objects of **C**
- **Morphisms**: Isomorphism classes of spans in **C**
- Composition defined by pullback:



- monoidal structure where $A \otimes B$ is the product in **C**, $A \times B$, and the unit is the terminal object I

To *linearize* a (finite) set, just take the free vector space on it, $\mathbb{C}^A$. Then there is a pair of linear maps associated to $f : A \to B$:

- $f^* : \mathbb{C}^B \to \mathbb{C}^A$, with $f^*(g) = g \circ f$
- $f_* : \mathbb{C}^A \to \mathbb{C}^B$, with $f_*(g)(b) = \sum_{f(a)=b} g(a)$

The first is just composition with f. The second is the map sending the vector δ_a to $\delta_{f(a)}$.

Using the standard inner product (such that the characteristic functions on A and B are orthonormal bases), these two maps are *linear adjoints*.

We can use this pair of adjoint maps to define a functor $L : Span_1(\mathbf{Set}) \to \mathbf{Vect}$:

Definition

For a set A, let $L(A) = \mathbb{C}^A$. Given a span $A \overset{s}{\leftarrow} X \overset{t}{\rightarrow} B$, define $L(X, s, t) = t_* \circ s^* : L(A) \to L(B)$.

So we have:

$$(L(X,s,t)(g))(b) = \sum_{t(x)=b} g(s(x))$$

This corresponds to multiplication by a matrix counting elements linking $a \in A$ and $b \in B$ ("sum over histories"):

$$L(X,s,t)_{a,b} = \#(s,t)^{-1}(a,b)$$

Composition by pullbacks in **Set** (*fibred products*) give an interpretation of matrix multiplication (counting composite paths):

$$A \times_C B = \coprod_{c \in C} f^{-1}(c) \times g^{-1}(c) = \{(a,b) | f(a) = g(b)\}$$

Theorem

This $L : Span_1(\mathbf{Set}) \rightarrow \mathbf{Vect}$ *is a monoidal functor.*

Note: The linear maps arising from $Span_1(\mathbf{Set})$ are all represented by matrices with *positive integer* entries. Groupoids (and $U(1)$-*groupoids*) will allow us to capture more of linear algebra.

Baez and Dolan described *groupoidification*, a way to extend the above to spans of groupoids.

Definition

A **groupoid** is a category in which all morphisms are invertible.

Groupoids describe "local symmetry":

Example

- Any set S can be seen as a groupoid with only identity morphisms
- Any group G is a groupoid with one object
- Given a set S with a group-action $G \times S \rightarrow S$ yields a transformation groupoid $S /\!\!/ G$ whose objects are elements of S; if $g(s) = s'$ then there is a morphism $g_s : s \rightarrow s'$
- The category $\mathbf{FinSet_0}$ of finite sets and bijections is a groupoid
- An orbifold or smooth stack is represented by a (smooth) groupoid

Definition

The **cardinality** of a groupoid $\mathbf{G}$ is

$$|\mathbf{G}| = \sum_{[g]\in\underline{\mathbf{G}}} \frac{1}{\#\,\mathrm{Aut}(g)}$$

where $\underline{\mathbf{G}}$ is the set of isomorphism classes of objects of $\mathbf{G}$. We call a groupoid **tame** if this sum converges.

This has the nice property that it "gets along with quotients":

Theorem (Baez, Dolan)

If S is a set with a G-action $G \times S \to S$, then

$$|S /\!\!/ G| = \frac{\#S}{\#G}$$

where $\#$ denotes ordinary set-cardinality.

Using groupoid cardinality instead of set-cardinality, one can extend L to a functor:

$$D : Span_1(\mathbf{Gpd}) \to \mathbf{Vect}$$

(Note: since **Gpd** is a 2-category, composition is by *weak* pullback.)
For objects: $D(G) = H_0(G)$ (the zeroth homology $\mathbb{C}^{\underline{G}}$).
For morphisms, we modify the formula for sets:

$$D(X,s,t)(g)([b]) = \sum_{[x] \in \underline{t^{-1}(b)}} \frac{\# \mathrm{Aut}(b)}{\# \mathrm{Aut}(x)} [g(s(x))]$$

These come from two maps f^* and f_* as before, which are adjoint with respect to an inner product such that $\langle [g_i], [g_j] \rangle = \frac{1}{\# \mathrm{Aut}(g_i)} \cdot \delta_{i,j}$. This the standard inner product on $D(G)$.

Definition

A **state** over a groupoid $\mathbf{G}$, in $Span_1(\mathbf{Gpd})$, is (up to isomorphism) a span:

$$\mathbf{1} \stackrel{!}{\leftarrow} X \stackrel{\Psi}{\rightarrow} \mathbf{G}$$

The **cardinality** of a state is given by $D(X, !, \Psi)$ seen as a vector:

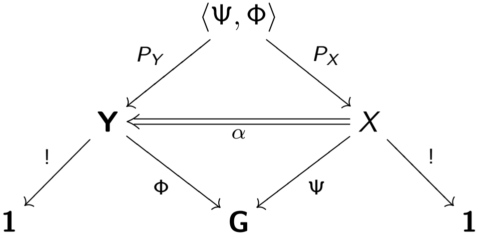
$$|\Psi| = \sum_{g \in \underline{\mathbf{G}}} |\Psi^{-1}(g)|[g] \in D(\mathbf{G})$$

where $|\Psi^{-1}(g)|$ is the groupoid cardinality of the *essential preimage* of g.

Example

In the case $\mathbf{G} = \mathbf{FinSet_0}$, a state is a (Baez-Dolan) "stuff type", which generalizes Joyal's "combinatorial species". Then Ψ is the "underlying set" functor, and objects of X are called "Ψ-stuffed finite sets" (or "Ψ-structured" when Ψ is faithful - i.e. when all morphisms in X are determined by those in $\mathbf{FinSet_0}$).

The composite of a state and costate is "just" a groupoid, as shown:

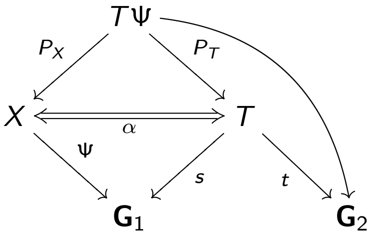


This defines an inner product on states: given two states, $\Psi : X \to \mathbf{G}$ and $\Phi : Y \to \mathbf{G}$, the inner product is a groupoid $\langle \Psi, \Phi \rangle$, given as a (weak) pullback.

Theorem (Baez, Dolan)

$$|\langle \Phi, \Psi \rangle| = \langle |\Phi|, |\Psi| \rangle$$

More generally, a span T in **Gpd** from $\mathbf{G}_1$ to $\mathbf{G}_2$ acts on a state Ψ over $\mathbf{G}_1$ by composition:



Theorem (Baez, Dolan)

$$|T\Psi| = |T||\Psi|$$

where $|T| = D(T, s, t)$ *is represented by the matrix with:*

$$|T|_{([a],[b])} = |(s,t)^{-1}(a,b)|$$

Idea: Groupoid cardinality gives an equivalence relation on groupoids, which is coarser than isomorphism. (Unlike sets, where cardinalities are isomorphism classes). Degroupoidification gives invariants (up to equivalence) for groupoids and spans, but they lose some information. We'll describe a richer invariant: a (weak) 2-functor

$$\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$$

where $Span_2(\mathbf{Gpd})$ is a 2-category of *spans of groupoids* and **2Vect** is the 2-category of *Kapranov-Voevodsky* 2-vector spaces:

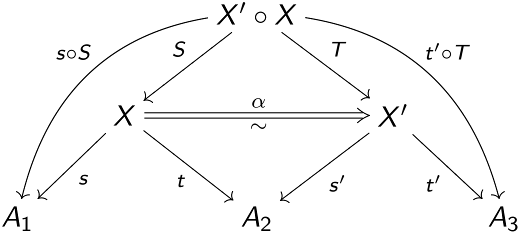
Definition

A **Kapranov–Voevodsky 2-vector space** is a $\mathbb{C}$-linear finitely semisimple additive category (one generated by simple objects x, where $\hom(x, x) \cong \mathbb{C}$). A **2-linear map** between 2-vector spaces is a $\mathbb{C}$-linear additive functor.

These, together with natural transformations between 2-linear maps, form a 2-category.

The bicategory $Span_2(\mathbf{Gpd})$ (similar for any 2-category with weak pullbacks) has:

- **Objects**: Groupoids
- **Morphisms**: Spans of groupoids
- Composition defined by *weak* pullback:



- **2-Morphisms** : isomorphism classes of *spans of* span maps
- monoidal structure from the product in **Gpd**

Note: This weak pullback of groupoids has objects (x, α, x'), where $\alpha : f(x) \to g(x')$, and its morphisms are commuting squares.

Theorem (Kapranov, Voevodsky)

Any 2-vector space is equivalent to $\mathbf{Vect^k}$ *(objects* k*-tuples of vector spaces, morphisms* k*-tuples of linear maps) for some* k*.*
Any 2-linear map $T : \mathbf{Vect}^k \rightarrow \mathbf{Vect}^l$ *is naturally isomorphic to a map of the form*

$$\begin{pmatrix} V_{1,1} & \dots & V_{1,k} \\ \vdots & & \vdots \\ V_{l,1} & \dots & V_{l,k} \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_k \end{pmatrix} = \begin{pmatrix} \bigoplus_{i=1}^k V_{1,i} \otimes W_i \\ \vdots \\ \bigoplus_{i=1}^k V_{l,i} \otimes W_i \end{pmatrix}$$

Any natural transformation can be written as a matrix of linear maps between the components.

Theorem

Any $\mathbb{C}$*-linear functor* $F : \mathbf{V_1} \rightarrow \mathbf{V_2}$ *between KV 2-vector spaces is necessarily additive and exact.*

Such a functor is a *2-linear map*.

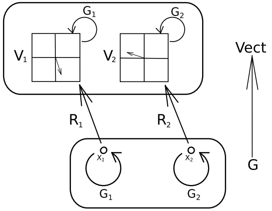
Lemma

If **G** *is an essentially finite groupoid, the functor category* $\Lambda(\mathbf{G}) = [\mathbf{G}, \mathbf{Vect}]$ *is a KV 2-vector space.*

Note: If the automorphism groups of (isomorphism classes of) objects of **G** are $G_1, \dots, G_n$, then we have

$$[\mathbf{G}, \mathbf{Vect}] \cong \prod_j \mathbf{Rep}(\mathbf{G_j})$$

So the "basis elements" (simple objects) in $[\mathbf{G}, \mathbf{Vect}]$ are labeled by $([g], V)$, where $[g] \in \underline{\mathbf{G}}$ and V an irreducible rep of $Aut(g)$.



Theorem (Morton)

If $\mathbf{X}$ *and* $\mathbf{B}$ *are essentially finite groupoids, a functor* $f : \mathbf{X} \to \mathbf{B}$ *gives two 2-linear maps:*

$$f^* : \Lambda(\mathbf{B}) \to \Lambda(\mathbf{X})$$

namely composition with f*, with* $f^*F = F \circ f$ *and*

$$f_* : \Lambda(\mathbf{X}) \to \Lambda(\mathbf{B})$$

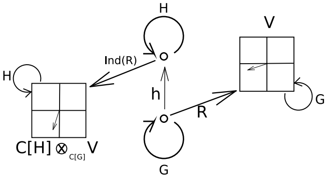
called "pushforward along f*". Furthermore,* f_* *is the two-sided adjoint to* f^* *(i.e. both left-adjoint and right-adjoint).*

In fact, the adjoint map

$$f_* : [\mathbf{X}, \mathbf{Vect}] \to [\mathbf{B}, \mathbf{Vect}]$$

to the composition acts on $F : \mathbf{X} \to \mathbf{Vect}$ to give the *induced representation*.

Given a group homomorphism $h : G \to H$, and a representation $R : G \to GL(V)$, there is an induced representation of H, namely $\mathbb{C}[H] \otimes_{\mathbb{C}[G]} V$:



For our functor of groupoids, $f : \mathbf{X} \to \mathbf{B}$, we can push forward a representation in the same way. If more than one object is sent to the same $b \in \mathbf{B}$, we get a direct sum of all their contributions:

$$f_*(F)(b) \cong \bigoplus_{f(x) \cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

(The direct sum is over the essential preimage of b in X.) This is a *Kan extension* of the functor F along f.

Definition

For a span of groupoids $X : A \rightarrow B$ in $Span_2(\mathbf{Gpd})$ define the 2-linear map:

$$\Lambda(X,s,t) = t_* \circ s^* : \Lambda(A) \longrightarrow \Lambda(B)$$

So then:

$$\Lambda(X,s,t)(F)(b) = \bigoplus_{t(x)=b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} (F \circ s)(x)$$

Picking basis elements $([a],V) \in \Lambda(A)$, and $([b],W) \in \Lambda(B)$, and using Frobenius reciprocity (i.e. the adjointness of our two 2-linear maps), we get that $\Lambda(X,s,t)$ is represented by the matrix:

$$\Lambda(X,s,t)_{([a],V),([b],W)} = \hom_{Rep(Aut(b))}(t_* \circ s^*(V), W)$$

$$\simeq \bigoplus_{[x] \in \underline{(s,t)^{-1}([a],[b])}} \hom_{Rep(\mathrm{Aut}(x))}(s^*(V), t^*(W))$$

The pull-push framework here is a special case of a general story about sheaves valued in different categories.
For *Set*-valued sheaves, this involves:

- Categories of sheaves, toposes
- Direct and inverse image functors f^*, f_* associated to continuous f
- Geometric Morphisms

For sheaves valued in Ab (abelian groups) or $R - Mod$ (modules) it involves:

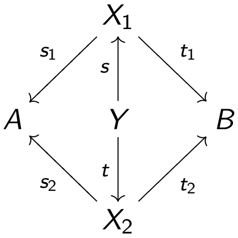
- Complexes of sheaves (resolutions)
- Derived categories (homotopy category)
- Grothendieck operations for derived categories

Things are simpler for us because:

- Vector spaces are flat (resolutions are trivial), and so
- All $\mathbb{C}$-linear functors are exact

However, for **Vect**-presheaves, we have something extra, because adjoints are two-sided:

The most general 2-morphism for us is a span between spans, $Y : X_1 \to X_2$ for $X_1, X_2 : A \to B$:



So define a natural transformation:

$$\Lambda(Y, s, t) : (t_1)_* \circ (s_1)^* \to (t_2)_* \circ (s_2)^*$$

Up to scale, this can be constructed using the unit and counit for the adjunctions between t^* and t_*, and between s^* and s_*, and can be seen as another pull-push construction.

Call the adjunctions in which f_* is left or right adjoint to f^* the *left and right adjunctions* respectively. These have f_* defined by left or right Kan extension, and are defined in each component using $hom_{\mathbb{C}[Aut(x)]}$ and $\otimes_{\mathbb{C}[Aut(x)]}$.

The *Nakayama isomorphism* is a canonical isomorphism between these (in particular: it defines an isomorphism even over base rings other than $\mathbb{C}$):

$$N : \bigoplus_{[x]|f(x)\cong b} \hom_{\mathbb{C}[Aut(x)]}(\mathbb{C}[Aut(b)], F(x)) \\ \rightarrow \bigoplus_{[x]|f(x)\cong b} \mathbb{C}[Aut(b)] \otimes_{\mathbb{C}[Aut(x)]} F(x)$$

given by the *exterior trace map* in each factor of the sum:

$$N : \bigoplus_{[x]|f(x)\cong b} \phi_x \mapsto \bigoplus_{[x]|f(x)\cong b} \frac{1}{\#Aut(x)} \sum_{g\in Aut(b)} g \otimes \phi_x(g^{-1})$$

We use the Nakayama isomorphism to identify left and right adjoints as follows...
For each map $f : X \rightarrow B$, there are units and counits for both the left and right adjunctions:

$$\begin{aligned}
\eta_L &: Id_{[X,\mathbf{Vect}]} \Longrightarrow f^* f_* \\
\epsilon_L &: f_* f^* \Longrightarrow Id_{[B,\mathbf{Vect}]} \\
\eta_R &: Id_{[B,\mathbf{Vect}]} \Longrightarrow f_* f^* \\
\epsilon_R &: f^* f_* \Longrightarrow Id_{[X,\mathbf{Vect}]}
\end{aligned}$$

Given a a span of span maps $Y : X_1 \rightarrow X_2$, define:

$$\Lambda(Y, s, t) = \epsilon_{L,t} \circ N^{-1} \circ \eta_{R,s}$$

In coordinates:

$$\Lambda(Y, s, t)_{[([a],V),([b],W)]} : \bigoplus_{[x_1]} \hom_{Rep(\mathrm{Aut}(x_1))}[s_1^*(V), t_1^*(W)] \\ \rightarrow \bigoplus_{[x_2]} \hom_{Rep(\mathrm{Aut}(x_2))}[s_2^*(V), t_2^*(W)]$$

For each pair $([x_1], [x_2])$, this gives a map

$$\hom[s_1^*(V), t_1^*(W)] \rightarrow \hom[s_2^*(V), t_2^*(W)]$$

given by a sum over $[y] \in (s, t)^{-1}([x_1], [x_2])$.

Note: now we have spaces of intertwiners between representations of the groups $\mathrm{Aut}(x_i)$. With D, we only had a copy of $\mathbb{C}$ for each $[x_i]$.

Pulling a representation F back by f^* gives a representation of a new group, on the same space. An intertwiner is a linear map on this space which commutes with the representation.

So if $\phi : s_1^*(V) \rightarrow t_1^*(W)$ is an intertwiner in $Rep(\mathrm{Aut}(x_1))$ then the same underlying linear map is also an intertwiner $s^*(\phi) : (s_1 \circ s)^*(V) \rightarrow (t_1 \circ s)^*(W)$ in $Rep(\mathrm{Aut}(y))$. However, for the pushforward along t, the corresponding linear operator behaves as follows:

For $\phi \in \hom[s_1^*(V), t_1^*(W)]$ we get:

$$\Lambda(Y, s, t)_{[([a],V),([b],W)]}|_{([x_1],[x_2])}(\phi) = \frac{|(s,t)^{-1}(x_1, x_2)|}{|\mathrm{Aut}(x_2)|} \sum_{g \in \mathrm{Aut}(x_2)} g \phi g^{-1}$$

(This uses the *essential* preimage $(s,t)^{-1}(x_1, x_2)$ as before.)

The *group average* here projects a linear map into the space of intertwiners.

Given all this, the main fact is:

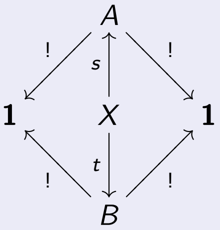
Theorem (Morton)

The process $\Lambda : Span_2(\mathbf{Gpd}) \rightarrow \mathbf{2Vect}$ is a weak 2-functor.

Furthermore:

Theorem (Morton)

Restricting to $\hom_{Span_2(\mathbf{Gpd})}(\mathbf{1}, \mathbf{1})$*:*



where $\mathbf{1}$ is the (terminal) groupoid with one object and one morphism, Λ on 2-morphisms is just the degroupoidification functor D.

For quantum mechanics, the classical configuration space S is usually not discrete.

Minimally, (S, μ) a measure space, and $L^2(S, \mu)$ is the Hilbert space for the corresponding quantum system. Interesting cases occur when S is a manifold and μ comes from a volume form.

Duplicating the above runs into some difficulties:

- Direct sums become direct integrals - which are not (co)limits
- Thus, push-forward is not just Kan extension of functors
- Topology is nontrivial, so must deal explicitly with sheaves
- Must work with infinite-dimensional vector spaces, which are not canonically iso. to double duals

Work on this is ongoing - but we'll describe some of the ingredients next.

Luckily, Hilbert spaces *are* canonically isomorphic to their double duals. So to start with:

Definition

If (X, μ) is a measurable space **Meas(X)** is the category with:

- Objects: *measurable fields of Hilbert spaces* on $(X, \mathcal{M})$: i.e. X-indexed families of Hilbert spaces $\mathcal{H}_x$ with a Hilbert space of *measurable sections*
- Morphisms: *measurable fields of bounded linear maps* between Hilbert spaces, $f_x : \mathcal{H}_x \rightarrow \mathcal{K}_x$ so that $||f||$ (the operator norm of f) is measurable.

This is the equivalent of a *measurable function*. Imposing L^2 condition gives a categorification of $L^2(X, \mu)$.

A measurable field of Hilbert spaces on X determines a *measurable sheaf* by direct integration: given a measurable $U \subset X$, this assigns

$$\int_U^{\oplus} d\mu(x)\mathcal{H}_x$$

where the direct integral is a Hilbert space of sections with inner product

$$\langle \phi, \psi \rangle = \int_U d\mu(x) \langle \phi_x, \psi_x \rangle$$

So this gives a *measurable sheaf of Hilbert spaces* in $MSh(X, \mu)$.
When X is a groupoid, a functor will define an *equivariant sheaf*: for each morphism $g : x \rightarrow x'$, the functor defines an isomorphism $\mathcal{H}_g : \mathcal{H}_x \rightarrow \mathcal{H}_{x'}$.
In particular, $\mathcal{H}_x$ carries a representation of the group $Aut(x)$.

Definition

A *disintegration* between two measure spaces consists of:

- A measurable function $f : (X, \mathcal{A}, \mu) \to (Y, \mathcal{B}, \nu)$
- A family $(X_y, \mathcal{A}_y, \mu_y)_{y \in Y}$ where:
  - $X_y = f^{-1}(y)$
  - $\mathcal{A}_y = \{A \cap X_y | A \in \mathcal{A}\}$
  - μ_y is a measure on X_y

satisfying some obvious properties.

Theorem (Wendt)

Given a disintegration $f : (X, \mu) \to (B, \nu)$*, there is an adjoint pair of functors*

$$MSh(X) \underset{f_*}{\overset{f^*}{\rightleftarrows}} MSh(Y)$$

Needed: An equivariant version of this theorem.

An equivalent of the external trace map and Nakayama isomorphism requires all groups around to have measures. When this all works out, instead of recovering groupoid cardinality, we should recover something like Weinstein's *volume of a differentiable stack*.

This requires that the groupoid be equipped with:

- A measure μ on the space of objects, and
- For each $x \in Ob(X)$, a measure ν_x on the space of morphisms into x, $t^{-1}(x)$

which transform in a specified way when passing from one groupoid representing the stack to another. (Here, "stack" = "equivalence class of groupoids")

Then the volume of the stack is:

$$vol(X) = \int_X d\mu(x) \Big(\int_{t^{-1}(x)} d\nu_x(g) \Big)^{-1}$$

Needed: Necessary and sufficient conditions on groupoids X to ensure this all works out. E.g. groupoids internal to *Meas* (some arise from constructions with compact Lie groups)

Generally, to extend both degroupoidification and 2-linearization to smooth/measured groupoids, we follow a similar construction, but:

- Finite product of $Rep(Aut(x_i)) \mapsto \mathbf{Meas}_{\mathcal{G}}(\mathcal{G}^{(0)})$
- Direct sum $\mapsto$ direct integral
- Counting measure $\mapsto$ measure on object space
- Groupoid cardinality $\mapsto$ volume of groupoid (cf. Weinstein)

The categorical/algebraic part of the construction is the same.